UDC 532.516

TWO KINDS OF INSTABILITY OF STATIONARY CONVECTIVE MOTION INDUCED BY INTERNAL HEAT SOURCES

PMM Vol. 37, №3, 1973, pp. 564-568

G. Z. GERSHUNI, E. M. ZHUKHOVITSKII and A. A. IAKIMOV
(Perm')

(Received April 30, 1972)

The stability of a stationary plane-parallel convective motion induced by heat sources uniformly distributed in a vertical layer of fluid is considered. Unlike in the earlier paper [1], thermal perturbations and their effect on the development of hydrodynamic perturbations are taken into consideration. A fairly substantial effect of thermal factors is established even at Prandtl number $P \sim 1$. With further increasing P these factors become determining: there occurs a change, the hydrodynamic instability of the counterflows becomes the instability of the type of thermal running waves. At considerable Prandtl numbers the Grashof number tends to zero according to the law $G_m \sim P^{-1/2}$.

Investigation of the stability of a plane-parallel convective flow between vertical parallel planes with respect to small normal perturbations reduces to the spectral problem of the amplitude of perturbation of the stream function φ and of temperature θ (see [1])

$$\begin{gathered}
\Delta^2\varphi - ikGH\varphi + \theta' = -\lambda\Delta\varphi \\
P^{-1}\Delta\theta + ikG\,(T_0'\varphi - v_0\theta) = -\lambda\theta \\
\varphi\,(\pm 1) = \varphi'\,(\pm 1) = \theta\,(\pm 1) = 0 \\
\Delta\varphi \equiv \varphi'' - k^2\varphi, \quad H\varphi \equiv v_0\Delta\varphi - v_0''\,\varphi \\
v_0 = {}^1/_{60}\,(1 - 6x^2 + 5x^4), \quad T_0 = 1 - x^2 \\
G = g\beta q h^5 / 2\nu^2, \quad P = \nu / \chi
\end{gathered} \tag{1}$$

All quantities in (1) are dimensionless, and v_0 and T_0 are, respectively, the unperturbed profiles of velocity and temperature, and k is the wave number. The decrement λ is the eigenvalue of the boundary value problem.

To solve the problem we use the Galerkin method, selecting the amplitudes of normal perturbations of the stream function and of temperature in the motionless fluid as the basic functions, which are defined by the boundary value problems

$$\Delta^2\varphi_i^{(0)} = -\mu_i^{(0)}\Delta\varphi_i^{(0)}, \qquad \varphi_i^{(0)}\,(\pm 1) = \varphi_i^{(0)'}\,(\pm 1) = 0$$

$$P^{-1}\Delta\theta_k^{(0)} = -\nu_k^{(0)}\theta_k^{(0)}, \qquad \theta_k^{(0)}\,(\pm 1) = 0$$

We represent the approximate solution of problem (1) in the form

$$\varphi = \sum_{i=0}^{N} a_i\varphi_i^{(0)}, \qquad \theta = \sum_{k=0}^{M} b_k\theta_k^{(0)} \tag{2}$$

The orthogonality conditions of Galerkin's method lead to a homogeneous system of linear equations for the coefficients a_i and b_k of the expansion. The perturbation decrements $\lambda = \lambda_r + i\lambda_i$ are found as the eigenvalues of the matrix of this system. The limit of stability of stationary motion relative to the corresponding perturbation is given by the condition $\lambda_r = 0$. The imaginary part of decrement λ_i is related to the perturbation phase velocity (in units of the stationary stream maximum velocity along the channel axis) by formula $c = (60 / kG)\, \lambda_i$. The eigenvalues λ were determined by diagonalization of the complex matrix on a computer with the use of the QR-algorithm.

Calculations show that the basic level of instability is related to perturbations of the "even" kind to which corresponds an even amplitude of the stream function and an odd amplitude of temperature perturbation with respect to the channel middle. These perturbations appear in the form of two vortex lines which are staggered at the boundaries of vertical streams [1].

In what follows we present only the results related to the basic instability level. The approximations (2) are derived with the use of basic functions $\varphi_i^{(0)}$ and $\theta_k^{(0)}$ of appropriate parity. Approximations containing six to fifteen functions in expansions of φ and θ were used in calculations. The comparison of results of various approximations shows that within the considered range of parameters these bases ensure a reasonable accuracy of determination of decrements and of the critical Grashof numbers.

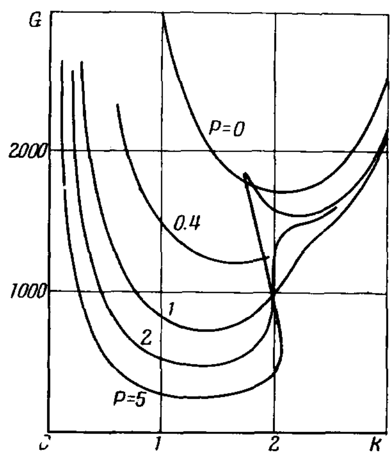

Fig. 1

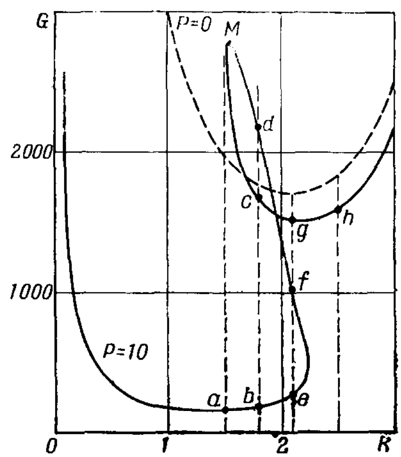

Fig. 2

Let us consider the obtained results. Neutral curves in the plane k, G are presented in Figs. 1 and 2 for some Prandtl numbers. The limiting curve $P = 0$ relates to the purely hydrodynamic approximation [1]. It can be seen that taking into consideration thermal factors considerably lowers the critical Grashof number with increasing Prandtl number. The absolute minimum of curve $G(k)$, which determines the stability limit, shifts toward lower k with increasing P, i.e. an increase of wavelength of the most dangerous perturbations takes place.

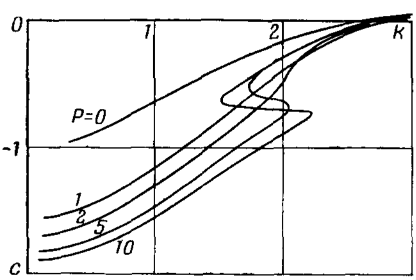

Fig. 3

The instability of flow with an even velocity profile is caused by running perturbations whose phase velocity is nonzero (see [1]). The effect of the wave number at various P

on the phase velocity of neutral perturbations is shown in Fig. 3. It is seen that in the wide interval of k, and in particular in the region of the neutral curve minimum, the phase velocity of perturbations is negative, i. e. the perturbations which disturb the stationary motion are running downward.

Summary data on the critical parameters which correspond to the absolute minimum of neutral curves are shown in Table 1. As shown by calculations which were made up to $P = 100$, the critical wave number $k_m \approx 1.4$, and the minimum Grashof number tends to zero according to the law $G_m = 488 / \sqrt{P}$.

This asymptotic behavior implies that at high P the critical difference of temperatures increases with viscosity as $\nu^{3/2}$. The limiting law definitely indicates that at high P the instability is caused by the buildup of running thermal waves in the stream (cf. [2, 3]).

Table 1

P	G_m	k_m	c_m
0	1720	2.05	−0.16
0.4	1219	1.65	−0.52
1	744	1.38	−0.87
2	470	1.35	−1.04
3	359	1.35	−1.12
5	259	1.35	−1.21
10	171	1.38	−1.29
20	115	1.40	−1.36

The dimensionless phase velocity of the most dangerous perturbations increases with increasing P . Thus at $P = 20$ it is more than one-and-a-half times greater than the velocity of downward currents (we recall that the maximum velocity $g\beta q h^4/120\nu$ of the convective stream ascending along the channel axis [1] is taken as the unit of phase velocity; in these units the maximum velocity of the downward currents is −0.8).

The considerable distortion of the neutral curve appearing with increasing Prandtl number is noticeable. The most significant decrease of the critical Grashof number occurs in the region of small k (the long-wave branch). The value $P = 5.7$ is to some extent critical, since at it the neutral curve has a cuspidal point (with coordinates $G = 1900$ and $k = 1.7$), while for $P > 5.7$ the neutral curve has a closed loop, clearly visible in Fig. 2 ($P = 10$). At high P line $G(k)$ essentially consists of two neutral curves which continuously pass from one to the other. The curve has two minima and it is possible to speak of two kinds of instability. One of the neutral curve branches (the short-wave one) is close to the corresponding curve for purely hydrodynamic perturbations (the "hydrodynamic instability mode"), while the other — the long-wave branch — which appears at fairly high P is essentially determined by thermal factors (the "thermal mode") (*). The most dangerous perturbation is related to the latter mode and corresponds to the absolute minimum of the neutral curve.

Thus a monotonic decrease of the critical Grashof number takes place with increasing Prandtl number. It is accompanied by a change of the instability mode from that of hydrodynamic at low P to the instability of the kind of increasing thermal waves at high P.

The change of the instability mode can be clearly observed by examining the spectra

*) The occurrence of the closed loop on the neutral curve at high P and the presence of two kinds of instability was recently noted, also, in the case of flow in the convective boundary layer at a heated vertical plate [4].

of characteristic perturbations. At low P, as well as in the limit case of $P = 0$, the instability is related to the lower even hydrodynamic level μ_0 (the notation introduced in [5] for the levels of convective perturbation spectra is used here and subsequently). A crowding of thermal levels takes place in the lower part of the spectrum with increasing Prandtl number and, beginning at $P = 1$ the instability at the long-wave branch of the neutral curve (small k) gradually passes to the lower thermal level ν_1. Further increase of P produces a widening of the wave number region in which the instability is associated with the level ν_1. At $P > 1.3$ this region extends to the whole neutral curve.

Formation of a loop on the neutral curve is associated with the interaction of two lower thermal levels ν_1 and ν_3. An illustration of this interaction is provided by the decrement spectra at $P = 10$ for various wave numbers. These spectra, shown in Fig. 4, relate to four vertical cross sections in plane k, G indicated in Fig. 2 by dash lines. Letters a, b, c, ... denote neutral points.

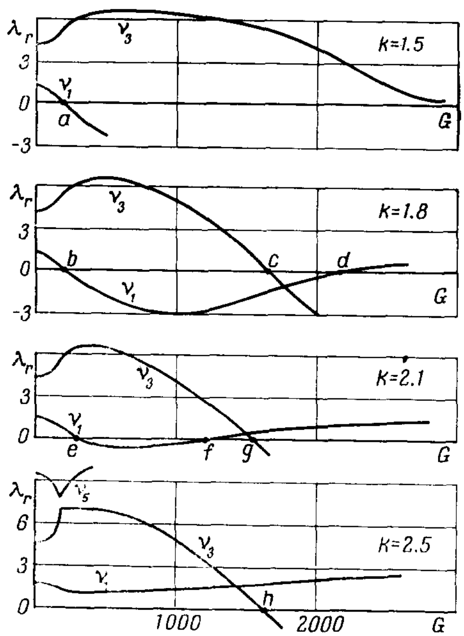

Fig. 4

Real parts of thermal levels ν_1 and ν_3 are shown in Fig.4. It will be seen that for $k = 1.5$ only the instability produced by the level ν_1 (neutral point a) exists to the left of the loop. The cross section $k = 1.8$ intersects the loop; in the interval $(b - d)$ we have instability with respect to perturbation ν_1, while to the right of point c it is with respect to ν_3; in region $(c - d)$ both perturbations increase. For $k = 2.1$ there is a ν_1-instability in the interval $(e - f)$ and a ν_3-instability to the right of point g. Finally, at cross section $k = 2.5$ we have only a ν_3-instability (neutral point h). Thus there exists at the neutral curve a point M at which separation of the two branches of the neutral curve takes place. Along the long-wave section instability is generated by the ν_1-perturbation and along the short-wave one it is generated by ν_3. The loop inner region corresponds to instability associated with both perturbations.

We note a particular peculiarity of spectra which, as far as the authors are aware, was not previously noted in investigations of hydrodynamic stability. This perculiarity consists of the appearance at certain definite values of parameters of a "complete intersection" of the complex decrements, at which the real and the imaginary parts of two intersecting modes coincide. An example of this is shown in Fig. 5 for $P = 3$ and $k = 1.95$ (a) and $k = 2.05$ (b), where the complete intersection of levels ν_1 and ν_3 takes place in the stability region ($\lambda_r > 0$). The formation of a loop on the neutral curve is, also, closely associated with the complete intersection of levels ν_1 and ν_3. The previously mentioned cuspidal point relates to those values of parameters P, G and k at which a complete intersection of decrements occurs at the neutral point $\lambda_r = 0$ (on the G-axis).

For $P > 5.7$ the complete intersection occurs in the instability region ($\lambda_r < 0$) for values of parameters corresponding to a point inside the loop. At $P = 10$ the complete intersection of levels ν_1 and ν_3 takes place in region $1.5 < k < 1.8$ (see Fig. 4). A situation close to the complete intersection of levels ν_3 and ν_5 ($k = 2.5$) is also seen in this figure.

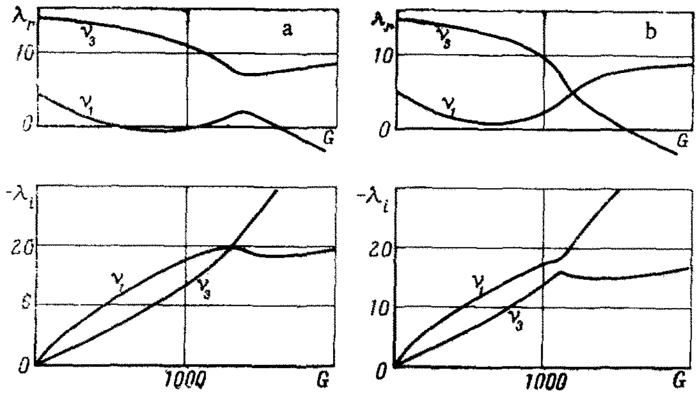

Fig. 5

Thus, as in the case of convective flow between planes heated to different tmperatures [2], the considered flow at high Prandtl numbers is unstable relative to thermal perturbations possessing a fairly high phase velocity (buildup of thermal waves in the stream). The difference is in that in the case of cubic profile flows different neutral curves correspond to instabilities of both kinds (factorization of the dispersion relationship owing to the oddness of the basic motion profile), with the thermal branch appearing at a certain Prandtl number P_*. In the case of convective flow induced by internal heat sources the new instability pattern is the result of continuous distortion of the single neutral curve which takes place with increasing Prandtl number.

So far instabilities of the kind of thermal wave buildup have been established in three cases: the flow between parallel planes heated to different temperatures; the boundary layer at a heated plate, and the flow induced by internal heat sources. It can be assumed that this form of instability is specific to any arbitrary convective flow.

BIBLIOGRAPHY

1. Gershuni, G. Z., Zhukhovitskii, E. M. and Iakimov, A. A., On the stability of steady convective motion generated by internal heat sources. PMM Vol. 34, №4, 1970

2. Birikh, R. V., Gershuni, G. Z., Zhukhovitskii, E. M. and Rudakov, R. N., On oscillatory instability of plane-parallel convective motion in a vertical channel. PMM Vol. 36, №4, 1972.

3. Gill, A. E. and Kirkham, C. C., A note on the stability of convection in a vertical slot. J. Fluid Mech., Vol. 42, pt. 1, 1970.

4. Hieber, C. A. and Gebhart, B., Stability of vertical natural convection boundary layers: some numerical solutions. J. Fluid Mech., Vol. 48, pt. 4, 1971.

5. Rudakov, R. N., On small perturbations of convective motion between vertical parallel planes. PMM Vol. 30, №2, 1966.

Translated by J. J. D.